
HYPERBOLICITY OF CYCLE SPACES AND AUTOMORPHISM GROUPS OF FLAG DOMAINS

by

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Abstract. — If G_0 is a real form of a complex semisimple Lie group G and Z is compact G -homogeneous projective algebraic manifold, then G_0 has only finitely many orbits on Z . Complex analytic properties of open G_0 -orbits D (flag domains) are studied. Schubert incidence-geometry is used to prove the Kobayashi hyperbolicity of certain cycle space components $\mathcal{C}_q(D)$. Using the hyperbolicity of $\mathcal{C}_q(D)$ and analyzing the action of $\text{Aut}_{\mathcal{O}}(D)$ on it, an exact description of $\text{Aut}_{\mathcal{O}}(D)$ is given. It is shown that, except in the easily understood case where D is holomorphically convex with a nontrivial Remmert reduction, it is a Lie group acting smoothly as a group of holomorphic transformations on D . With very few exceptions it is just G_0 .

1. Introduction

Throughout this paper Z denotes a compact projective algebraic manifold which is homogeneous with respect to a holomorphic action of a connected complex semi-simple group G . Such manifolds are alternatively described by $Z = G/Q$ where Q is a (complex) parabolic subgroup of G . A Lie subgroup G_0 of G is said to be a *real form* of G whenever its Lie algebra \mathfrak{g}_0 is defined as the fixed point space of an antilinear involution $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$. Since G_0 has only finitely many orbits in Z ([W]), it has at least one and usually many open orbits D . Manifolds of the type Z are called *flag manifolds* and the open G_0 -orbits D are called *flag domains*. In all applications standard decompositions allow us

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to reduce to the case where G_0 is simple and therefore we assume this. Note that if G_0 is complex, being embedded in G by an antiholomorphic map, then $G = G_0 \times G_0$. This is the only situation where it may not be assumed that G is simple.

It is possible that $D = Z$. This occurs when G_0 is compact and there are several exceptional noncompact cases as well ([O1, O2], see Proposition 5.2.1 in [FHW]). From the point of view of the present work, these cases are not of interest. Thus we not only assume that G_0 is a simple noncompact real form of G but also that D is not compact.

Let us fix a maximal compact subgroup K_0 of G_0 and denote by K its complexification in G . A basic first example of Matsuki-duality states that K_0 has exactly one orbit C_0 in D which is a complex submanifold ([W]). It is in fact the only K -orbit in Z which is contained in D .

Fix $q := \dim_{\mathbb{C}}(C_0)$ and recall that a q -cycle in Z is a formal linear combination

$$C = n_1 X_1 + \dots + n_m X_m$$

where the coefficients n_i are positive integers and the X_i are compact, irreducible, q -dimensional subvarieties of Z . The set of all such cycles has the natural structure of complex space ([B], see [FHW] §7.4 for the properties needed here). In the particular case where Z is projective, it is the Chow variety. Its components are themselves projective algebraic and the G -action on these components is algebraic. In many situations we regard C_0 as the cycle $1.C_0$ in the cycle space and refer to it as the *base cycle*. Since the cycle space is smooth at C_0 ([FHW] Theorem 18.6.1), the notion of the irreducible component of the cycle space of Z which contains C_0 makes sense. This is denoted by $\mathcal{C}_q(Z)$ and

$$\mathcal{C}_q(D) := \{C \in \mathcal{C}_q(Z); C \subset D\}.$$

It should be remarked that, although it is known that $\mathcal{C}_q(D)$ is smooth at the base point C_0 , it is an open question as to whether or not it is everywhere smooth.

The following is the first main result of this paper.

Theorem 1.1. — *The complex space $\mathcal{C}_q(D)$ is Kobayashi-hyperbolic.*

The proof of this result is established using the *Schubert incidence geometry* defined by special Borel subgroups of G . For this, recall that a Borel subgroup B has only finitely many orbits \mathcal{O} in Z . Each is abstractly algebraically isomorphic to some $\mathbb{C}^{m(\mathcal{O})}$. These are referred to as *Schubert cells*. Since the B -action

on Z is algebraic, every such orbit is Zariski open in its closure. We denote such a closure by S and write $S = \mathcal{O} \dot{\cup} E$, where E is the union of the (lower-dimensional) B -orbits on the boundary of \mathcal{O} . The varieties S , which in fact often have singularities along E , are called *Schubert varieties*. Observe that the CW-decomposition of Z which is defined by the Schubert varieties only has cells in even (real) dimensions. Thus they freely generate the integral homology $H_*(Z, \mathbb{Z})$.

If $G_0 = K_0 A_0 N_0$ is an Iwasawa-decomposition and the Borel group B contains the solvable subgroup $A_0 N_0$, we refer to it as being an Iwasawa-Borel subgroup. These subgroups can be viewed in the flag manifold G/B of all Borel subgroups as the points on the (unique) closed G_0 -orbit.

Now consider the set $\mathcal{S}(B)$ of $(n-q)$ -dimensional Schubert varieties of an Iwasawa-Borel subgroup B . It follows from Poincaré duality that at least some of the S in $\mathcal{S}(B)$ have nonempty (set-theoretic) intersection with C_0 . The following is the essential ingredient for a number of considerations in this context (see Theorem 9.1.1 in [FHW]).

Theorem 1.2. — *If $S = \mathcal{O} \dot{\cup} E \in \mathcal{S}(B)$ is an $(n-q)$ -dimensional Schubert variety of an Iwasawa-Borel subgroup of G and $S \cap C_0 \neq \emptyset$, then $S \cap C_0$ is a finite subset $\{z_1, \dots, z_m\}$ which is contained in \mathcal{O} . At each z_i the intersection $S \cap C_0$ is transversal. The orbit $A_0 N_0 \cdot z_i =: \Sigma_i$ is open in S and closed in D ; in particular $S \cap D \subset \mathcal{O}$. Furthermore, $\Sigma_i \cap C_0 = \{z_i\}$.*

If S is as in the above Theorem, then we consider the *incidence variety*

$$D_S := \{C \in \mathcal{C}_q(Z); C \cap E \neq \emptyset\}.$$

It is important that, for appropriate multiplicities of its components, this is a Cartier divisor. (see [FHW], §7.4, and the Appendix of [HS]). Theorem 1.1 is proved by considering the maps given by the linear systems defined by the incidence divisors D_S . These are pieced together in such a way that after paying the small price of a finite map, $\mathcal{C}_q(D)$ is realized as an open subset of a hyperbolic domain in a product of projective spaces. The hyperbolicity of this latter domain is proved by applying the classically proven fact that the complement the union of $2n + 1$ hyperplanes in general position in \mathbb{P}_n is hyperbolic.

The method of Schubert incidence geometry has played a role in much of our work over the last years. In particular, in [H1] we used this method to prove the hyperbolicity of the group-theoretically defined cycle space \mathcal{M}_D which is the connected component containing the base cycle of the intersection $G \cdot C_0 \cap$

$\mathcal{C}_q(D)$. The proof here of hyperbolicity in the case of the full cycle space $\mathcal{C}_q(D)$ is substantially more involved.

It should be remarked that, except for a certain Hermitian cases which are well-understood, \mathcal{M}_D only depends on G_0 , i.e., it doesn't vary as the G -flag manifold Z and the G_0 -flag domain D vary (see [FHW]). On the other hand, $\mathcal{C}_q(D)$ varies (sometimes wildly) as Z and D vary (Part IV of [FHW]). Thus it would seem that the cycle spaces $\mathcal{C}_q(D)$ might provide a wide range of interesting Kobayashi-hyperbolic spaces which could be useful for holomorphically realizing G_0 -representations.

Theorem 1.1 is one of the essential ingredients in the proof of the following second main result of this paper (see §5, in particular Theorem 5.1).

Theorem 1.3. — *Unless D is holomorphically convex with nontrivial Remmert reduction, the automorphism group $\text{Aut}_{\mathcal{O}}(D)$ is a Lie group acting smoothly on D as a group of holomorphic transformations.*

This leads to a detailed description of the connected component $\text{Aut}_{\mathcal{O}}(D)^{\circ}$ for every flag domain D . In order to prove Theorem 1.3 we characterize holomorphic convexity by a cycle condition and show that if this condition is not satisfied, then the natural map $\text{Aut}_{\mathcal{O}}(D) \rightarrow \text{Aut}_{\mathcal{O}}(\mathcal{C}_q(D))$ is finite-fibered and is a local homeomorphism onto its closed image. The desired results then follow from the fact that the automorphism group of a hyperbolic complex space is a Lie group.

As is indicated in the statement of Theorem 1.3 the exception to the finite-dimensionality occurs in a setting which is optimal from the point of view of complex analysis, namely where D is holomorphically convex but not Stein. This can only occur in the case where G_0 is of Hermitian type and even in that case it is very rare. The Remmert reduction $D \rightarrow \widehat{D}$, which in this case is induced from a fibration $Z \rightarrow \widehat{Z}$ of the ambient flag manifolds, is a G_0 -equivariant homogeneous fibration onto the G_0 -Hermitian symmetric space of noncompact type embedded in its compact dual. This situation can be characterized in a number of ways (see §3).

In §6 it is shown that unless $Z = G/Q$ and Q is a maximal parabolic subgroup, or equivalently $b_2(Z) = 1$, it follows that $\tilde{G}_0 := \text{Aut}_{\mathcal{O}}(D) = G_0$. Even when $b_2(Z) = 1$ there are very few exceptional cases. A key point for the classification of these cases is that if \tilde{G}_0 properly contains G_0 , then the complexification \tilde{G} also acts transitively on Z . Therefore we are in a position to apply Onishchik's

classification of that situation, i.e., that where two different complex simple groups act transitively on Z .

2. Hyperbolicity of the cycle space

This section is devoted to the proof of Theorem 1.1. For this we at first consider an Iwasawa-decomposition $G_0 = K_0 A_0 N_0$ and fix an Iwasawa-Borel subgroup B which contains $A_0 N_0$. Let $\mathcal{S}_{C_0}(B)$ be the set of $(n-q)$ -dimensional B -Schubert varieties which have nonempty intersection with the base cycle C_0 . For $S = \mathcal{O} \cup E \in \mathcal{S}_{C_0}(B)$ let D be a Cartier divisor with support on the incidence variety defined by E and let L be the associated line bundle on $\mathcal{C}_q(Z)$ with section s defining D .

We may choose G to be simply-connected so that L has the unique structure of a G -bundle with s being a B -eigenvector in the finite-dimensional G -representation space $\Gamma(\mathcal{C}_q(Z), L)$. Since this representation may not be irreducible, we let V be the irreducible subrepresentation which contains s as a B -highest weight vector and consider the G -equivariant meromorphic map

$$\varphi : \mathcal{C}_q(Z) \rightarrow \mathbb{P}(V^*)$$

defined by $\varphi(x) := \{s; s(x) = 0\}$.

Observe that since E is contained in the complement of D , the restriction of φ to $\mathcal{C}_q(D)$ is base point free. Furthermore, the image $\varphi(\mathcal{C}_q(D))$ is contained in the complement $\mathbb{P}(V^*) \setminus H_0$ of the hyperplane which corresponds to the section s .

Note that since H_0 is $A_0 N_0$ -invariant, the family $\mathcal{F} := \{gH_0\}_{g \in G_0}$ is compact. As a consequence the union $\cup_{H \in \mathcal{F}} H$ is compact and its open complement Ω contains the image $\varphi(\mathcal{C}_q(D))$. In [H1] (see also [FH, FHW]) we utilize the fact that the d -dimensional G -representation on V^* is irreducible to prove the following fact.

Theorem 2.1. — *The family \mathcal{F} contains $2d - 1$ hyperplanes H_1, \dots, H_{2d-1} which are in general position in $\mathbb{P}(V^*)$. In particular, Ω is Kobayashi-hyperbolic.*

The technical work for the proof of Theorem 1.1 amounts to carrying out the above construction for each of the Schubert varieties in $\mathcal{S}(B)$ and then to show

that the restriction to $\mathcal{C}_q(D)$ of the product of the resulting maps is finite-fibered. For a precise statement let S_1, \dots, S_m be a list of the Schubert varieties in $\mathcal{S}_{C_0}(B)$ and V_1, \dots, V_m be the irreducible G -representations with highest weight vectors s_1, \dots, s_m obtained as above. Let $\varphi_j : \mathcal{C}_q(Z) \rightarrow \mathbb{P}(V_j^*)$ be the associated meromorphic maps.

Proposition 2.2. — *The restriction of the meromorphic map*

$$\psi := \varphi_1 \times \dots \times \varphi_m : \mathcal{C}_q(Z) \rightarrow \mathbb{P}(V_1^*) \times \dots \times \mathbb{P}(V_m^*)$$

to $\mathcal{C}_q(D)$ is holomorphic and finite-fibered.

The proof of this Proposition, which goes by assuming the contrary and deriving a contradiction, requires some notational preliminaries. For this let $\widehat{\mathcal{C}_q(Z)}$ be the normalized graph of ψ and

$$\widehat{\psi} : \widehat{\mathcal{C}_q(Z)} \rightarrow \mathbb{P}(V_1^*) \times \dots \times \mathbb{P}(V_m^*)$$

be the resulting holomorphic map.

We suppose to the contrary of the claim in Proposition 2.2 that \widehat{Y} is a compact complex curve in some $\widehat{\psi}$ -fiber and that \widehat{Y} has nonempty intersection with the (biholomorphic) lift of $\mathcal{C}_q(D)$ in $\widehat{\mathcal{C}_q(Z)}$. Under that assumption let Y denote the projection of \widehat{Y} in $\mathcal{C}_q(Z)$, \mathfrak{X}_Y the preimage of Y in the universal family $\mathfrak{X} \rightarrow \mathcal{C}_q(Z)$ and let X be the $(q+1)$ -dimensional image of \mathfrak{X}_Y in Z .

Now consider the intersection of X with any one of the Schubert varieties in $\mathcal{S}_{C_0}(B)$ which for simplicity we denote by $S = \mathcal{O} \cup E$. Since $X \cap D \neq \emptyset$ and every such S intersects every cycle (transversally) in D in only finitely many points, it follows that $S \cap X$ contains a 1-dimensional component which has nonempty intersection with D . Since $\mathcal{O} \cong \mathbb{C}^{m(\mathcal{O})}$, the intersection with $X \cap E$ is nonempty.

Lemma 2.3. — *Every $C \in Y$ with $C \cap E \neq \emptyset$ is contained in the base point set B_ψ of the meromorphic map defined by the Schubert variety S .*

Proof. — Let s be the section of the bundle L which is the B -highest weight vector in the representation space V defined by the incidence divisor D_S . Since E is contained in the complement of D , for those cycles $\tilde{C} \in Y \cap \mathcal{C}_q(D)$ it follows that $s(\tilde{C}) \neq 0$. But $s(C) = 0$. Thus if C were not a base point, it would follow that $\varphi(C) \neq \varphi(\tilde{C})$. This is contrary to \widehat{Y} being contained in a $\widehat{\psi}$ -fiber. \square

Proof of Proposition 2.2. In order to complete the proof, we will produce a Schubert variety $S \in \mathcal{S}_{C_0}(B)$ so that if $C \in Y$ as in the above Lemma, then C is *not* a base point of the associated meromorphic map. For this we consider the closure $\text{cl}(G.C)$ of the G -orbit of C in the full cycle space $\mathcal{C}_q(Z)$ and let C_1 be a B -fixed point in a closed G -orbit in $\text{bd}(G.C)$. Then C_1 is the union of finitely many q -dimensional Schubert varieties. Recall that Poincaré duality in Z is realized by the intersection pairing of B -Schubert varieties with Schubert varieties of the opposite Borel subgroup B^* . In particular, there is an $(n-q)$ -dimensional B^* -Schubert variety $S_1 = \mathcal{O} \cup E$ which has nonempty finite intersection with C_1 with $C_1 \cap E = \emptyset$.

Since convergence at the level of cycles (regarded as points) is the same thing as Hausdorff convergence, it follows that if gC is sufficiently near C_1 in the cycle space, then S_1 has nonempty finite intersection with gC with $gC \cap E$ likewise being empty. Consequently there is a proper algebraic subset A of G with the property that if $g \notin A$, then C has finite nonempty intersection with gS_1 and $C \cap gE = \emptyset$.

Now S_1 is not a B -Schubert variety, but if we choose B carefully, the opposite Borel B^* is also an Iwasawa-Borel subgroup which is K_0 -conjugate to B . For the sake of completeness let us explain this choice in some detail. Let σ be the involution which defines the maximal compact subalgebra \mathfrak{g}_u and which commutes with the involution τ which defines \mathfrak{g}_0 so that $\sigma|_{\mathfrak{g}_0} =: \theta$ is the Cartan involution. This defines the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ with \mathfrak{a}_0 defined to be a maximal Abelian subspace of \mathfrak{p}_0 . Let \mathfrak{m}_0 be the centralizer of \mathfrak{a}_0 in \mathfrak{k}_0 and choose a toral subalgebra \mathfrak{t}_0 in \mathfrak{m}_0 so that $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ is a real form of the σ -invariant Cartan subalgebra $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ of \mathfrak{g} . Finally define a system of positive roots for \mathfrak{g} with respect to \mathfrak{h} so that the direct sum \mathfrak{n}_0^+ of the positive restricted root spaces defines our Iwasawa decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0^+$. Now let \mathfrak{b} be the Borel subalgebra in the minimal parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ which consists of \mathfrak{h} together with the positive root spaces. This is a semidirect sum $\mathfrak{b} = \mathfrak{b}_m \ltimes (\mathfrak{a} \oplus \mathfrak{n}^+)$ where \mathfrak{b}_m is a Borel subalgebra of \mathfrak{m} . The opposite Borel algebra \mathfrak{b}^* which consists of \mathfrak{h} together with the negative root spaces is of the form $\mathfrak{b}^* = \mathfrak{b}_m^* \ltimes (\mathfrak{a} \oplus \mathfrak{n}^-)$ and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0^-$ is the corresponding Iwasawa-decomposition.

Since K_0 acts transitively on the Iwasawa-decompositions with compact summand \mathfrak{k}_0 , for some $k_0 \in K_0$ it follows that $k_0 S_1 = S$, where $S \in \mathcal{S}_{C_0}(B)$. As a result, for $g \notin Ak_0^{-1}$ it follows that $C \cap gS$ is nonempty and finite, and

$C \cap gE = \emptyset$. But the section $g(s)$ defined by the incidence divisor of gE is linearly equivalent to that defined by S and $C \cap gE = \emptyset$ translates to $g(s)(C) \neq 0$. Consequently C is not a base point of the meromorphic map φ defined by S . \square

We now come to the final steps in the proof of the hyperbolicity of the cycle space.

Proof of Theorem 1.1. Let W be the image of holomorphic map $\widehat{\psi} : \widehat{\mathcal{C}_q(Z)} \rightarrow \mathbb{P}(V_1^*) \times \dots \times \mathbb{P}(V_m^*)$ and $\widehat{\psi}_{Stein} : \widehat{W} \rightarrow W$ its Stein factorization. Note that Proposition 2.2 can be reformulated to state that $\mathcal{C}_q(D)$ is naturally embedded in \widehat{W} . Recall that $\mathbb{P}(V_j^*)$ contains an open subset Ω_j which is the complement of the union of the hyperplanes in a family \mathcal{F}_j , which is Kobayashi-hyperbolic and which contains the image of the restriction of φ_j to $\mathcal{C}_q(D)$. Thus the intersection Ω of the product $\Omega_1 \times \dots \times \Omega_m$ with W has the same properties: It contains the image of $\mathcal{C}_q(D)$ (now regarded as a subset of \widehat{W}) and is Kobayashi-hyperbolic. To complete the proof observe that the restriction of $\widehat{\psi}_{Stein}$ to the preimage $\widehat{\Omega}$ of Ω is a proper finite-fibered holomorphic map. Since Ω is hyperbolic, it follows that $\widehat{\Omega}$ is likewise hyperbolic ([K], Proposition 3.2.11). Finally, since $\mathcal{C}_q(D)$ has been realized as an open subset of $\widehat{\Omega}$, it is immediate that it is hyperbolic as well. \square

3. Cycle reduction

Here we discuss two equivariant holomorphic equivalence relations which are defined by the cycles in D . The first has been introduced in [H2], but our point of view here is different and for the sake of completeness we provide the essential details.

Cycle connectivity. — Let us say that two points $p, q \in D$ are *connected by cycles* if there are finitely many cycles $C_1, \dots, C_N \in \mathcal{C}_q(D)$ so that $p \in C_1, q \in C_N$ and the chain $C_1 \cup \dots \cup C_N$ is connected. This notion defines an equivalence relation where two points are equivalent whenever they are connected by a chain of cycles. It is G_0 -equivariant in the sense that for all $g \in G_0$

$$p \sim q \Leftrightarrow g(p) \sim g(q).$$

Let p_0 be a base point with the property that $K_0.p_0 = C_0$ is the base cycle. If p is connected to p_0 by a chain $C_1 \cup \dots \cup C_N$ to p_0 with $p_0 \in C_1$ and $p \in C_N$ and

$k_0 \in K_0$, then $k_0(p)$ is connected to p_0 by the chain $C_0 \cup k_0(C_1) \cup \dots \cup k_0(C_N)$. Thus the equivalence class $[p_0]$ is K_0 -invariant.

Now write $D = G_0/H_0$ where H_0 is the isotropy subgroup of the point p_0 . The reduction $D \rightarrow \widehat{D}/\sim$ defined by cycle connectivity is G_0 -equivariant and therefore it is a homogeneous fibration $G_0/H_0 \rightarrow G_0/I_0$. If $I_0 = H_0$, in other words if any two points can be connected by cycles, we say that D is *cycle connected*.

Proposition 3.1. — *If D is not cycle connected, then $I_0 = K_0$, i.e., the base \widehat{D} of the cycle reduction is the associated G_0 -Hermitian symmetric space.*

Proof. — Above it was shown that K_0 is contained in the stabilizer of the equivalence class $[p_0]$. Recall that K_0 is not only a maximal compact subgroup of G_0 but is in fact a maximal subgroup. Thus the stabilizer of $[p_0]$ is either the full group G_0 or is K_0 . In the former case D is cycle connected and in the latter case $[p_0] = C_0$. \square

Observe that in the case where D is not cycle connected there is an open neighborhood U of the identity in the complex G -isotropy subgroup Q at p_0 which stabilizes the equivalence class $[p_0] = C_0$. Since U generates Q , it follows that $[p_0]$ is Q -invariant. Define \widehat{Q} as the stabilizer of $[p_0]$ in G and note that it contains both K and Q .

Proposition 3.2. — *If D is not cycle connected, then the restriction R of the fibration $Z = G/Q \rightarrow G/\widehat{Q} = \widehat{Z}$ to the flag domain D is the quotient $G_0/H_0 = D \rightarrow \widehat{D} = G_0/K_0$ defined by cycle connectivity equivalence in the strong sense that the fibers of $D \rightarrow \widehat{D}$ are the same as the fibers of $Z \rightarrow \widehat{Z}$ over points of \widehat{D} .*

Proof. — The restriction R maps D onto a flag domain D_1 whose G_0 -isotropy at its base point contains K_0 . Since D is not cycle connected, the neutral fiber of $G/Q = Z \rightarrow \widehat{Z} = G/\widehat{Q}$ is the cycle C_0 ; in particular, \widehat{Z} is not just a point. Since the maximal subgroup K_0 is the stabilizer of C_0 in G_0 , it follows that $D_1 = \widehat{D}$. \square

Let us summarize the results of this paragraph.

Corollary 3.3. — *Under the assumption that D is not a Hermitian symmetric space of noncompact type, the following are equivalent:*

1. *The flag domain D is not cycle connected with reduction $G_0/H_0 = D \rightarrow \widehat{D} = G_0/K_0$ defined by cycle connectivity equivalence.*
2. *The flag domain D is holomorphically convex with nontrivial Remmert reduction. In particular, the base \widehat{D} of its Remmert reduction $G_0/H_0 = D \rightarrow \widehat{D} = G_0/K_0$ is a Hermitian symmetric space \widehat{D} which is embedded as a G_0 -orbit in its compact dual \widehat{Z} and its neutral fiber is C_0 .*

Proof. — If D is holomorphically convex, then the stated properties concerning its Remmert reduction are known ([W], see also [FHW], p. 63). Thus, in order to show that 2.) implies 1.) we must only note that, since its base is Stein, the Remmert reduction maps every chain of cycles in D to a point. Therefore D is not cycle connected. The direction 1.) implies 2.) follows immediately from that fact that the fiber of the reduction defined by cycle connectivity equivalence is the compact variety C_0 and the base is the Stein Hermitian symmetric space. \square

The group $\text{Aut}_{\mathcal{O}}(D)$ in the holomorphically convex case. — One of the purposes of this article is to describe in detail the automorphism groups of flag domains. The case where D is not cycle connected is special in that $\text{Aut}_{\mathcal{O}}(D)$ is not a Lie group. Let us now describe this group.

Assume that D is not cycle connected with Remmert reduction $D \rightarrow \widehat{D}$. Note that since the Hermitian symmetric space \widehat{D} is Stein and retractible, the bundle $D \rightarrow \widehat{D}$ is holomorphically trivial. Thus D is biholomorphically equivalent to the product $C_0 \times \widehat{D}$ of the base cycle and a Hermitian symmetric space of non-compact type. The latter can be realized as a bounded domain and therefore its automorphism group is a (known) Lie group acting smoothly on D as a group of holomorphic transformations.

Since $\text{Aut}_{\mathcal{O}}(C_0)$ is a complex Lie group, we may consider the space $\text{Hol}(\widehat{D}, \text{Aut}_{\mathcal{O}}(C_0))$ of holomorphic maps of the base of the Remmert reduction to this group. Every $\varphi \in \text{Hol}(\widehat{D}, \text{Aut}_{\mathcal{O}}(C_0))$ defines a holomorphic automorphism of the product $C_0 \times \widehat{D}$, namely by $(z, w) \mapsto (\varphi(w)(z), w)$. Conversely, the projection $\pi : C_0 \times \widehat{D} \rightarrow \widehat{D}$ is equivariant with respect to $\text{Aut}_{\mathcal{O}}(C_0 \times \widehat{D})$ and its ineffectivity on \widehat{D} is exactly $\text{Hol}(\widehat{D}, \text{Aut}_{\mathcal{O}}(C_0))$. Thus, although it is quite large, $\text{Aut}_{\mathcal{O}}(D)$ can be concretely described.

Proposition 3.4. — *If D is not cycle connected with cycle equivalence reduction given by $D \rightarrow \hat{D}$, then*

$$\mathrm{Aut}_{\mathcal{O}}(D) \cong \mathrm{Hol}(\hat{D}, \mathrm{Aut}_{\mathcal{O}}(C_0)) \rtimes \mathrm{Aut}_{\mathcal{O}}(\hat{D}).$$

The following remark which concludes this paragraph will be of use in the next section.

Proposition 3.5. — *If $Z \rightarrow Z_1$ is a G -equivariant fibration which induces a mapping $D \rightarrow D_1$ of flag domains and D is cycle connected, then D_1 is cycle connected.*

Proof. — If D_1 is not cycle connected, then it is holomorphically convex; in particular, $\mathcal{O}(D_1) \not\cong \mathbb{C}$. Thus $\mathcal{O}(D) \not\cong \mathbb{C}$ and as a result D is likewise holomorphically convex ([W]). \square

Remark. It should be noted that if we had defined the cycle connectivity equivalence relation using only the cycles in \mathcal{M}_D , i.e., those of the form $g(C_0)$ for $g \in G$, then the entire discussion above could have been repeated. In particular, D is cycle connected in this sense if and only if it is cycle connected in the weaker sense where we use cycles from the full cycle space $\mathcal{C}_q(D)$. \square

Cycle separation. — The *cycle separation* equivalence relation is defined by $p \sim q$ if and only if every cycle in $\mathcal{C}_q(D)$ which contains p also contains q and vice versa. In other words, if \mathcal{F}_p denotes the family of cycles which contain p , then $p \sim q$ if and only if $\mathcal{F}_p = \mathcal{F}_q$. Since $g(\mathcal{F}_p) = \mathcal{F}_{g(p)}$ for all $g \in \mathrm{Aut}_{\mathcal{O}}(D)$, the equivalence relation is $\mathrm{Aut}_{\mathcal{O}}(D)$ -equivariant. In particular, it is G_0 -equivariant. Therefore, if $D = G_0/H_0$, where H_0 is the isotropy group of the neutral point p_0 in D , then the reduction $D \rightarrow D/\sim =: \tilde{D}$ is a G_0 -fibration $G_0/H_0 = D \rightarrow \tilde{D} = G_0/I_0$ where I_0 is the stabilizer of the equivalence class $[p_0]$.

For $p \in D$ define $\mathcal{I}_p := \cap_{C \in \mathcal{F}_p} C$ and observe that

$$[p_0] = \cap_{p \in \mathcal{I}_{p_0}} \mathcal{I}_p.$$

The following is an immediate consequence of this description.

Proposition 3.6. — *The cycle separation equivalence classes $[p]$ are closed compact complex analytic subvarieties of D . In particular, if the base cycle C_0 is defined by $K_0 \cdot p_0$, then $[p_0]$ is contained in C_0 .*

Using this proposition we now show that reduction by cycle separation can be extended to a G -equivariant fibration $Z \rightarrow \tilde{Z}$ of the ambient projective space.

Lemma 3.7. — *The equivalence class $[p_0]$ is invariant with respect to the complex G -isotropy group $G_{p_0} = Q$.*

Proof. — If U is a sufficiently small open neighborhood of the identity in Q , then $U.[p_0] \subset D$. If $u(p) \notin [p_0]$ for some $u \in U$ and some $p \in [p_0]$, then there is either a cycle C_1 which contains p_0 and not $u(p)$ or a cycle C_2 which contains $u(p)$ and not p_0 . In the first case, contrary to $p \in [p_0]$ the cycle $u^{-1}(C_1)$ contains p_0 but not p . In the second case, C_2 is not of the form $u(C)$ for some cycle containing p_0 , because $p_0 \notin C_2$. Thus $u^{-1}(C_2)$ is a cycle containing p but not p_0 and as above this is contrary to $p \in [p_0]$. Consequently $u(p) \in [p_0]$ for all $u \in U$ and $p \in [p_0]$ and, since U generates Q as a group, it follows that $[p_0]$ is Q -invariant. \square

Therefore the stabilizer \tilde{Q} of $[p_0]$ in G is a parabolic subgroup of G which defines a holomorphic fibration $G/Q = Z \rightarrow \tilde{Z} := G/\tilde{Q}$. Since $\tilde{Q} \supset I_0$ and I_0 acts transitively on $[p_0]$, it follows that $\tilde{Q}/Q = [p_0]$. This information can be summarized as follows.

Proposition 3.8. — *The restriction of the holomorphic fibration $\pi : G/Q = Z \rightarrow \tilde{Z} = G/\tilde{Q}$ to D is the reduction $\pi_0 : G_0/H_0 = D \rightarrow \tilde{D} = G/I_0$ of D by cycle separation equivalence. For $p \in D$ the fibers of the two maps agree: $\pi^{-1}(\pi(p)) = \pi_0^{-1}(\pi_0(p))$. In particular, \tilde{D} is a flag domain in \tilde{Z} .*

Let us say that that *cycles separate the points* of a flag domain D if $D = \tilde{D}$. The following shows that one actually gains something by reducing to \tilde{D} .

Proposition 3.9. — *For every flag domain D the cycles separate the points of the base \tilde{D} of the cycle separation equivalence relation.*

The next lemma is the essential ingredient for the proof of this Proposition. For this we let $\tilde{q} := \dim(\tilde{C}_0)$ and denote by π^* the map from the full space of \tilde{q} -cycles in \tilde{Z} to the full space of q -cycles in Z which is induced from the fibration $\pi : Z \rightarrow \tilde{Z}$.

Lemma 3.10. — *The restriction of π^* to the irreducible component $\mathcal{C}_{\tilde{q}}(\tilde{Z})$ is a biholomorphic map $\pi^* : \mathcal{C}_{\tilde{q}}(\tilde{Z}) \rightarrow \mathcal{C}_q(Z)$. In particular, $\pi_0^* : \mathcal{C}_{\tilde{q}}(\tilde{D}) \rightarrow \mathcal{C}_q(D)$ is also biholomorphic.*

Proof. — The map π^* is holomorphic and injective and therefore its image lies in some irreducible component of the full cycle space of Z . Since $\pi^*(\tilde{C}_0) = C_0 \in \mathcal{C}_q(Z)$, its image is in $\mathcal{C}_q(Z)$. Now consider the map π_* from supports of cycles in $\mathcal{C}_q(Z)$ to compact analytic subsets in \tilde{Z} . Let us say that a cycle $C \in \mathcal{C}_q(Z)$ is π -saturated if its image is \tilde{q} -dimensional. Clearly the set of saturated cycles is closed. To see that it is also open, suppose C is saturated, but that there is a sequence C_n of nonsaturated cycles converging to C . It follows that $\pi(C_n)$ converges to $\pi(C)$ in the Hausdorff topology. But this is impossible, because $\dim(\pi(C_n)) > \dim(\pi(C))$. Since C_0 is saturated, it follows that every $C \in \mathcal{C}_q(Z)$ is saturated and therefore π_* maps $\mathcal{C}_q(Z)$ to an irreducible subvariety of the full space of \tilde{q} -cycles in \tilde{Z} . Since $\pi_*(C_0) = \tilde{C}_0$, this subvariety is contained in $\mathcal{C}_{\tilde{q}}(\tilde{Z})$. Therefore π^* is invertible with inverse π_* . The biholomorphicity of π_0^* follows immediately. \square

Proof of Proposition 3.9. Observe that if $\tilde{p} \in \tilde{D}$, then by the Lemma the family $\mathcal{F}_{\tilde{p}}$ of cycles containing \tilde{p} is mapped biholomorphically by π_0^* to \mathcal{F}_p for any $p \in \pi^{-1}(\tilde{p})$. If \tilde{p} and \tilde{q} are different points in \tilde{D} and p and q are corresponding points in D , then $\mathcal{F}_p \neq \mathcal{F}_q$. Thus $\mathcal{F}_{\tilde{p}} \neq \mathcal{F}_{\tilde{q}}$ and consequently \tilde{p} and \tilde{q} are separated by some cycle in \tilde{D} . \square

Since $\mathcal{C}_q(D)$ is Kobayashi-hyperbolic with $\text{Aut}_{\mathcal{O}}(\mathcal{C}_q(D))$ being a Lie group acting smoothly by holomorphic transformations, the following formal observation is important for our study of $\text{Aut}(D)$.

Proposition 3.11. — *If cycles separate points in D , then the canonically defined homomorphism $\text{Aut}_{\mathcal{O}}(D) \rightarrow \text{Aut}_{\mathcal{O}}(\mathcal{C}_q(D))$ is injective.*

Proof. — If $g \in \text{Aut}(D)$ acts as the identity on the cycle space, then $g(\mathcal{F}_q) = \mathcal{F}_q$ and consequently $g(\mathcal{I}_q) = \mathcal{I}_q$ for all $q \in D$. Recall that $[p]$ is the intersection of such sets and $[p] = \{p\}$ in the case where cycles separate points. Thus if cycles separate points and g acts trivially on the cycle space, it follows that g acts trivially on D . \square

As a consequence it follows that for an arbitrary flag domain D the ineffectivity of the $\text{Aut}_{\mathcal{O}}(D)$ -action on $\mathcal{C}_q(D)$ is the same as the ineffectivity of its action on the base of the reduction $D \rightarrow \tilde{D}$ defined by cycle separation. In the next section we show that, unless D is holomorphically convex with nontrivial Remmert reduction, the latter ineffectivity is finite.

4. Ineffectivity of the action on the cycle space

In this section it will be assumed that the flag domain at hand is cycle connected, or equivalently that D is not holomorphically convex. Recall that this is the same as assuming that $\mathcal{O}(D) \cong \mathbb{C}$. As above $D \rightarrow \tilde{D}$ denotes the reduction by cycle separation.

Our goal here is to prove that the ineffectivity I of the $\text{Aut}_{\mathcal{O}}(D)$ -action on the cycle space $\mathcal{C}_{\tilde{q}}(\tilde{D})$ is a finite (perhaps trivial) group. For this we must slightly refine our considerations for the reduction by cycle separation. This is done as follows.

The reduction by cycle separation $D \rightarrow \tilde{D}$ is constructed using all cycles in $\mathcal{C}_q(D)$, because for the purposes of analyzing $\text{Aut}_{\mathcal{O}}(D)$ we require it to be $\text{Aut}_{\mathcal{O}}(D)$ -equivariant. On the other hand, by doing this we lose control of the geometric nature of the cycles, because they are not necessarily orbits. Having paid this price, we now reduce \tilde{D} by cycle separation using only those cycles from $\mathcal{M}_{\tilde{D}}$. Thus we obtain a sequence

$$D \rightarrow \tilde{D} \rightarrow \tilde{D}_{res}$$

of G_0 -homogeneous fibrations. The notation \tilde{D}_{res} refers to the fact that we have *restricted* to the smaller set of cycles $\mathcal{M}_{\tilde{D}}$. Note that since $\mathcal{M}_{\tilde{D}}$ is naturally isomorphic to \mathcal{M}_D , we may view $D \rightarrow \tilde{D}_{res}$ as reduction by cycle separation using \mathcal{M}_D .

The methods which are used for proving the basic properties of $D \rightarrow \tilde{D}$ may be used for $D \rightarrow \tilde{D}_{res}$. For example, the sequence of homogeneous fibrations above are saturated restrictions of fibrations

$$Z \rightarrow \tilde{Z} \rightarrow \tilde{Z}_{res}.$$

In this case, however, we only know that $\pi_{res} : D \rightarrow \tilde{D}_{res}$ defines a G_0 -equivariant biholomorphic map from $\mathcal{M}_{D_{res}}$ to \mathcal{M}_D .

Now we come to the main point: Although $D \rightarrow \tilde{D}_{res}$ is not necessarily $\text{Aut}_{\mathcal{O}}(D)$ -equivariant, it is I -invariant! This is just due to the fact that $D \rightarrow \tilde{D}_{res}$ can be defined by continuing the I -invariant map $D \rightarrow \tilde{D}$ to $D \rightarrow \tilde{D} \rightarrow \tilde{D}_{res}$. Using this we will show that I is a (finite-dimensional) Lie group acting smoothly on D as a group of holomorphic transformations.

The Lie group structure of I . — Our work here makes strong use of the proposition below. It will be applied to the associated fibration $\pi : C \rightarrow C_{res}$

of a cycle C in \mathcal{M}_D to $C_{res} \in \mathcal{M}_{D_{res}}$ which is defined by the restriction of the reduction $D \rightarrow \tilde{D}_{res}$. Note that if $C = g(C_0)$, then C and C_{res} are homogeneous with respect to $\hat{G} := gKg^{-1}$ and π is \hat{G} -equivariant.

Proposition 4.1. — *Let $\pi : \hat{G}/Q_1 = Z_1 \rightarrow Z_2 = \hat{G}/Q_2$ be a homogeneous fibration of flag manifolds. Let I_1 be a Lie group acting effectively on Z_1 as a closed subgroup of $\text{Aut}_{\mathcal{O}}(Z_1)$ which stabilizes every π -fiber $\pi^{-1}(z_2) =: F$. Then I_1 acts effectively on every π -fiber.*

Proof. — We may replace I_1 by the (algebraic) subgroup I of the algebraic group $\text{Aut}_{\mathcal{O}}(X)$ which stabilizes the π -fibers. Now $\text{Aut}_{\mathcal{O}}(X)^{\circ}$ is semisimple and I° is a product of its simple factors. Thus $Z_1 = A \times B$, where A is the corresponding product of irreducible factors of Z_1 . Consequently A is contained in every π -fiber and as a result I° acts effectively on every π -fiber as well. Finally, since I is an algebraic group I° is semisimple, any normal subgroup Γ of I is finite. Therefore, if Γ is the ineffectivity of the I -action on a π -fiber F , at any point $z \in F$ we may faithfully linearize the Γ -action on the tangent space $T_z Z_1$ to produce a local submanifold Δ complementary to F at z which is Γ -invariant and where Γ acts faithfully. This is contrary to Γ stabilizing all π -fibers. \square

The following is the main result of this section.

Theorem 4.2. — *If D is a flag domain which is not holomorphically convex, then the ineffectivity I of the $\text{Aut}(D)$ -action on the base of the reduction $D \rightarrow \tilde{D}$ defined by cycle separation is a Lie group acting smoothly as a group of holomorphic transformations on D .*

Before giving the proof, let us note that by the basic result of Bochner and Montgomery ([BM], Theorem 4) it is enough to show that I is a Lie group acting *continuously* on D . Thus we begin by looking a bit carefully at the topological properties of I . For this let $\nu : \mathfrak{X} \rightarrow \mathcal{M}_D$ be the universal family where $\mathfrak{X} = \{(z, C) \in D \times \mathcal{M}_D; z \in C\}$ and ν is the projection on the second factor. Again we emphasize that $\text{Aut}_{\mathcal{O}}(D)$ may not act on \mathfrak{X} but I does, because it stabilizes every cycle in \mathcal{M}_D . Since ν is proper and I -invariant, we may apply the theorem of Arzela-Ascoli-Montel to prove the following fact.

Proposition 4.3. — *Every sequence $\{g_n\} \in I$ has a subsequence which converges in I .*

Proof. — The application of the theorem of Arzela-Ascoli-Montel shows that after passing to a subsequence $\{g_n\}$ converges to $g \in \text{Aut}_{\mathcal{O}}(\mathfrak{X})$. But this implies that $g_n|_C \rightarrow g|_C$ for every $C \in \mathcal{M}_D$. Now view g_n as a sequence in $\text{Aut}_{\mathcal{O}}(D)$. Given $x \in C \subset D$ the image point $g_n(x)$ doesn't depend on C . Thus the map $g : D \rightarrow D$ is well-defined by $g(x) := g|_C(x)$. In other words g descends via the map $\mathfrak{X} \rightarrow D$ to an automorphism of D . \square

Given $C \in \mathcal{M}_D$ we let $R_C : I \rightarrow \text{Aut}_{\mathcal{O}}(C)$, $g \mapsto g|_C$, the restriction map. The following is an immediate consequence of the the above compactness statement and the continuity of R_C .

Corollary 4.4. — *The restriction map $R_C : I \rightarrow \text{Aut}_{\mathcal{O}}(C)$ is a closed map.*

Proof of Theorem 4.2. Consider the I -invariant fibration $\pi : D \rightarrow \tilde{D}_{res}$ defined by cycle separation with cycles in \mathcal{M}_D . If $C = g(C_0) \in \mathcal{M}_D$, then the restriction π_C of π to C maps C to a cycle \tilde{C}_{res} in $\mathcal{M}_{\tilde{D}_{res}}$. Both cycles are homogeneous with respect to $\hat{G} := gKg^{-1}$ and $\pi_C : C \rightarrow \tilde{C}_{res}$ is \hat{G} -equivariant. Let I_C be the image of I in $\text{Aut}_{\mathcal{O}}(C)$ and $I_C(F)$ the image of I_C in the automorphism group of a given π_C -fiber F . The above Propostion shows that the map $I_C \rightarrow I_C(F)$ defined by restriction is an isomorphism.

Since D is not holomorphically convex, we know that it is cycle connected with respect to cycles in \mathcal{M}_D . Thus \tilde{D}_{res} is cycle connected with respect to cycles in $\mathcal{M}_{\tilde{D}_{res}}$. Given two points \tilde{p} and \tilde{q} in \tilde{D}_{res} we connect them with a chain $\tilde{C}_{res}^1 \cup \dots \cup \tilde{C}_{res}^m$ with intersection points \tilde{p}_j so that $\tilde{p}_0 = \tilde{p}$ and $\tilde{p}_n = \tilde{q}$. Let F_j be the π -fibers over the \tilde{p}_j . The above argument shows that if $g \in I$ is in the kernel of the restriction map $I \rightarrow I_{C_j}$, then it is in the kernel of the restriction map $I \rightarrow I_{C_j}(F_j)$. Again applying the above argument, this implies that g is in the kernel of $I \rightarrow I_{C^{j+1}}$. Using the fact that D is cycle connected, this shows that if g is in the kernel of the restriction map $I \rightarrow I_C(F)$ for some cycle C and some fiber F , then $g = \text{Id}$, i.e., the homomorphism $R_C : I \rightarrow I_C(F)$ is an injective continuous homomorphism onto a subgroup of the Lie group $\text{Aut}_{\mathcal{O}}(F)$. It follows from the above Lemma that R_C is a homeomorphism onto a closed subgroup $I_C(F)$ of $\text{Aut}_{\mathcal{O}}(F)$. Since $I_C(F)$ is a Lie group, it follows follows that I is a Lie group (see Theorem 1.1 of Chapter VIII in [H]). Since it is a (closed) subgroup $\text{Aut}_{\mathcal{O}}(D)$, the action of I on D is continuous. \square

We have now shown that ineffectivity I is a Lie group. It is normalized by G_0 and, since both I and G_0 are subgroups of the topological group $\text{Aut}_{\mathcal{O}}(D)$, the action of G_0 on I by conjugation is continuous. By the results in Chapter IX

of [H], in particular Theorem 3.1, the semidirect product $I \rtimes G_0$ is a Lie group which is acting continuously on D . The following is therefore a consequence of the Theorem of Bochner and Montgomery ([BM]).

Corollary 4.5. — *The Lie group $I \rtimes G_0$ acts smoothly as a group of holomorphic transformations of D .*

Finiteness of the ineffectivity. — After all this discussion about the ineffectivity I we are now in a position to show that it is essentially nonexistent: Except when D is holomorphically convex with a nontrivial Remmert reduction, I is finite! The following is the remaining tool needed for proving this.

Proposition 4.6. — *If \tilde{G}_0 is a connected Lie group acting effectively on D as a group of holomorphic transformations and $\tilde{G}_0 \supset G_0$, then the action $\tilde{G}_0 \times D \rightarrow D$ extends to a smooth action $\tilde{G}_0 \times Z \rightarrow Z$ by holomorphic transformations.*

Proof. — Let $n := \dim_{\mathbb{C}}(D)$ and consider the space \tilde{V} of sections of the anticanonical bundle of D spanned by elements of the form $\xi_1 \wedge \dots \wedge \xi_n$ where the ξ_j are arbitrary holomorphic vector fields defined by the local \tilde{G} -action on D . Define the \mathfrak{g} -anticanonical map $\varphi : D \rightarrow \mathbb{P}(\tilde{V}^*)$ by $p \mapsto \{s \in \tilde{V}; s(p) = 0\}$. Since \tilde{G}_0 acts on \tilde{V} and D is homogeneous, φ is a \tilde{G}_0 -equivariant holomorphic map (see [HO] for other basic properties of this map).

Let p_0 be the neutral point in D and $z_0 := \varphi(p_0)$ its image in $\mathbb{P}(\tilde{V}^*)$. Now D is embedded in the G -flag manifold Z whose anticanonical bundle is very ample. In fact the \mathfrak{g} -anticanonical map extends to $Z = G/Q$ as the Tits fibration $G/Q \rightarrow G/N$, where N is the normalizer of Q in G . Since $N = Q$ (Borel's Normalizer Theorem), the \mathfrak{g} -anticanonical map of D is an embedding. Since the vector space V which is defined by limiting the ξ_j to fields defined by \mathfrak{g} is contained in \tilde{V} , it is therefore immediate that φ is an embedding.

Thus we may regard D as \tilde{G}_0 -orbit $\tilde{G}_0.z_0$ in $\mathbb{P}(\tilde{V}^*)$ and define \tilde{G} to be the smallest complex Lie subgroup containing the group \tilde{G}_0 in $\mathrm{GL}(\tilde{V})$. Hence D can be regarded as the open \tilde{G}_0 -orbit $\tilde{G}_0.z_0$ in the complex orbit $\tilde{G}.z_0$.

After this lengthy background, we consider the complex orbit $G.z_0$ which is an open subset in $\tilde{G}.z_0$. Since φ is an embedding, the isotropy algebra \mathfrak{g}_{z_0} is just the Lie algebra \mathfrak{q} . Thus the Lie algebra of G_{z_0} is \mathfrak{q} and therefore $G_{z_0} = Q$, in particular $G.z_0 = G/Q = Z$ is compact. Since $G.z_0$ is open in $\tilde{G}.z_0$ and both groups are connected, it follows that $\tilde{G}.z_0 = G.z_0 = Z$ as desired. \square

Theorem 4.7. — *Unless D is holomorphically convex with nontrivial Remmert reduction, the ineffectivity I of the action of $\text{Aut}_{\mathcal{O}}(D)$ on the cycle space $\mathcal{C}_q(D)$ is finite.*

Proof. — Note that the connected component I° is normalized by G_0 and Corollary 4.5 implies that $\tilde{G}_0 := I^\circ \rtimes G_0$ is a connected Lie group which acts smoothly as a group of holomorphic transformations on D . The ineffectivity of this action is contained in the center of G_0 and is therefore discrete. By definition G_0 acts on Z and it follows from the above Proposition that \tilde{G}_0 acts on Z . By definition the complexification G acts transitively on Z and therefore the smallest complex subgroup \tilde{G} of $\text{Aut}_{\mathcal{O}}(Z)$ which contains \tilde{G}_0 also acts transitively.

We claim that \tilde{G} properly contains G . To see this note that since \tilde{G} acts transitively on Z , it is semisimple. Thus the Lie algebra \mathfrak{i} is semisimple and \mathfrak{g}_0 is the direct sum $\mathfrak{i} \oplus \mathfrak{g}_0$. As a result \mathfrak{g} is not simple. Thus if $\mathfrak{g} = \mathfrak{g}_0$, it follows that \mathfrak{g} is not simple and therefore \mathfrak{g}_0 is (abstractly) a simple complex Lie algebra embedded antiholomorphically in its complexification $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_0$. But if $\mathfrak{g} = \mathfrak{g}_0$, then the image of \mathfrak{g}_0 in \mathfrak{g} centralizes \mathfrak{i} . On the other hand this image has trivial centralizer. Thus \mathfrak{g} properly contains \mathfrak{g}_0 .

As a result we are in the situation where two complex Lie groups G and \tilde{G} act transitively on Z . Since \tilde{G} properly contains G , it follows that $Z = G/Q$ where Q is a *maximal* parabolic subgroup ([O1, O2]). But this situation arises when $D \rightarrow \tilde{D}$ is a nontrivial reduction by cycle separation. Since this reduction is the restriction of a fibration $Z \rightarrow \tilde{Z}$ of the ambient flag manifold, Q is *not* a maximal parabolic subgroup of G . Thus \mathfrak{i} is trivial and therefore I is discrete. But since it is normalized, and therefore centralized by G_0 , it is finite. \square

5. Lie group properties

Let us begin by stating the main result of this section. For this we recall that if a flag domain D is not cycle connected, then it is holomorphically convex and the reduction $D \rightarrow \hat{D}$ by cycle connectivity equivalence is its Remmert reduction. The base \hat{D} is a Hermitian symmetric space of noncompact type and is therefore realizable as a bounded Stein domain. In particular, if $D = \hat{D}$, then $\text{Aut}_{\mathcal{O}}(D)$ is a Lie group acting smoothly on D as a group of holomorphic transformations. If D is not cycle connected and $D \neq \hat{D}$, then $\text{Aut}_{\mathcal{O}}(D)$ is infinite-dimensional, but nevertheless easily describable (see 3.4). Here we shall prove that if D is cycle connected, then $\text{Aut}_{\mathcal{O}}(D)$ is a Lie group acting smoothly on

D as a group of holomorphic transformations. This then completes the proof of the following result.

Theorem 5.1. — *If D is a flag domain with reduction $D \rightarrow \widehat{D}$ by cycle connectivity, then one of the following three cases holds:*

1. *D is not cycle connected, in which case it is holomorphically convex. The reduction $D \rightarrow \widehat{D}$ is the Remmert reduction of D has positive-dimensional fiber F and positive-dimensional base \widehat{D} which is a Hermitian symmetric space of noncompact type. The group of automorphisms, which can be described by*

$$\mathrm{Aut}_{\mathcal{O}}(D) \cong \mathrm{Hol}(\widehat{D}, \mathrm{Aut}_{\mathcal{O}}(F)) \rtimes \mathrm{Aut}_{\mathcal{O}}(\widehat{D}),$$

is infinite-dimensional.

2. *D is a Hermitian symmetric space of noncompact type, i.e., $D = \widehat{D}$, and $\mathrm{Aut}_{\mathcal{O}}(D)$ is an Lie group acting smoothly on D as a group of holomorphic transformations.*
3. *D is cycle connected, i.e., \widehat{D} is just a point, and $\mathrm{Aut}_{\mathcal{O}}(D)$ is a Lie group acting smoothly on D as a group of holomorphic transformations.*

Let us now complete the proof of this theorem. We must only handle the case where D is cycle connected or equivalently where $\mathcal{O}(D) \cong \mathbb{C}$. The proof uses the fact that $\mathrm{Aut}_{\mathcal{O}}(D)$ can essentially be identified with a subgroup of $\mathrm{Aut}_{\mathcal{O}}(\mathcal{C}_q(D))$. The following is a first step in this direction.

Proposition 5.2. — *If D is any flag domain, then the canonically defined continuous homomorphism*

$$\iota : \mathrm{Aut}_{\mathcal{O}}(D) \rightarrow \mathrm{Aut}_{\mathcal{O}}(\mathcal{C}_q(D))$$

is a closed map.

Proof. — Let $\mathfrak{X} \subset D \times \mathcal{C}_q(D)$ be the universal family of cycles with its projections $\mu : \mathfrak{X} \rightarrow D$ and $\nu : \mathfrak{X} \rightarrow \mathcal{C}_q(D)$. Suppose F is a closed subset of $\mathrm{Aut}_{\mathcal{O}}(D)$ and that $g_n = \iota(h_n)$ defines a sequence in $\iota(F)$ which converges to $g \in \mathrm{Aut}_{\mathcal{O}}(\mathcal{C}_q(D))$. Observe that the g_n act on \mathfrak{X} and denote them there by \tilde{g}_n . Since $\nu : \mathfrak{X} \rightarrow \mathcal{C}_q(D)$ is proper and $g_n \rightarrow g$, it follows that the sequence $\{\tilde{g}_n\}$ is equibounded. It therefore follows from the theorem of Arzela-Ascoli that after going to a subsequence $\tilde{g}_{n_k} \rightarrow \tilde{g}$, where \tilde{g} is a lift of g to \mathfrak{X} . Now the \tilde{g}_n are lifts of the h_n and since $\tilde{g}_n \rightarrow \tilde{g}$, the automorphism \tilde{g} of \mathfrak{X} descends to an automorphism h of D with $h_n \rightarrow h$. Since F is closed, $h \in F$ and, since ι is continuous, $\iota(h) = g$. Thus $\iota(F)$ is likewise closed. \square

Corollary 5.3. — *If cycles separate points in D , then*

$$\iota : \text{Aut}_{\mathcal{O}}(D) \rightarrow \text{Aut}_{\mathcal{O}}(\mathcal{C}_q(D))$$

is an injective homeomorphism onto a closed subgroup of $\text{Aut}_{\mathcal{O}}(\mathcal{C}_q(D))$.

Proof. — It remains to prove the injectivity. However, this is the content of Proposition 3.11. \square

Using the Theorem of Bochner and Montgomery ([BM], Theorem 4) the above corollary shows that Theorem 5.1 holds in the case where $D = \tilde{D}$. In the case where the fiber of the reduction $D \rightarrow \tilde{D}$ is positive-dimensional we make use of the following result.

Proposition 5.4. — *The continuous homomorphism*

$$\pi_* : \text{Aut}_{\mathcal{O}}(D) \rightarrow \text{Aut}_{\mathcal{O}}(\tilde{D})$$

which is induced by cycle separation $\pi : D \rightarrow \tilde{D}$ is a closed map.

Proof. — Recall that π induces a biholomorphic map $\mathcal{C}_{\tilde{q}}(\tilde{D}) \rightarrow \mathcal{C}_q(D)$. If F is a closed subset of $\text{Aut}_{\mathcal{O}}(D)$ and $g_n = \pi_*(h_n) \in \pi_*(F)$ defines a sequence which converges to $g \in \text{Aut}_{\mathcal{O}}(\tilde{D})$, then the associated sequence of automorphisms of $\mathcal{C}_{\tilde{q}}(\tilde{D})$ converges to the automorphism of the cycle space associated to g . Thus the sequence $\{\iota(h_n)\}$ on the cycle space $\mathcal{C}_q(D)$ which is associated to $\{h_n\}$ converges and the result follows from Proposition 5.2. \square

As a result we know that the image of π_* is a closed subgroup of $\text{Aut}_{\mathcal{O}}(\tilde{D})$ and is therefore a Lie subgroup. Recall that the $\text{Ker}(\pi_*) = I$ is finite. Thus the map $\pi_* : \text{Aut}_{\mathcal{O}}(D) \rightarrow \text{Aut}_{\mathcal{O}}(\tilde{D})$ is a topological covering map of a Lie group. As a result $\text{Aut}_{\mathcal{O}}(D)$ is homeomorphic to a Lie group and again by the Theorem of Bochner and Montgomery it follows that $\text{Aut}_{\mathcal{O}}(D)$ is a Lie group acting smoothly on D as a group of holomorphic transformations. This completes the proof of Theorem 5.1.

6. Detailed description of $\text{Aut}_{\mathcal{O}}(D)$

Recall that our initial setting is that of a simple real form G_0 of a complex semisimple group G acting on a G -flag manifold $Z = G/Q$. Except in the case where D holomorphically convex with nontrivial Remmert reduction, where $\text{Aut}_{\mathcal{O}}(D)$ is a certain precisely described infinite-dimensional group, $\text{Aut}_{\mathcal{O}}(D)$

is a Lie group acting smoothly on D by holomorphic transformations. In this section we restrict to the latter case and give a more detailed description of $\text{Aut}_{\mathcal{O}}(D)$. Here is a first step in that direction.

Proposition 6.1. — *If Q is not maximal, i.e., if the Betti number $b_2(Z)$ is at least two, then $\text{Aut}_{\mathcal{O}}(D)^{\circ} = G_0$.*

Proof. — If $\text{Aut}_{\mathcal{O}}(D)^{\circ} =: \tilde{G}_0$, then by Proposition 4.6 the action of \tilde{G}_0 extends to a smooth action $\tilde{G}_0 \times Z \rightarrow Z$ of holomorphic transformations on Z . Since $\tilde{G}_0 \supset G_0$, the smallest complex Lie group \tilde{G} in $\text{Aut}_{\mathcal{O}}(Z)$ which contains \tilde{G}_0 also acts transitively on Z . As we observed in the proof of Theorem 4.7, if \tilde{G}_0 properly contains G_0 , then \tilde{G} properly contains G . Thus it follows from the work of Onishchik ([O1, O2]) that Q is maximal. \square

Corollary 6.2. — *If G is defined to be $\text{Aut}_{\mathcal{O}}(Z)^{\circ}$ and D is not holomorphically convex with nontrivial Remmert reduction, then $\text{Aut}_{\mathcal{O}}(D)^{\circ} = G_0$.*

Exceptional cases. — If G is not the connected component \tilde{G} of the full automorphism group $\text{Aut}_{\mathcal{O}}(Z)$ then, as we just noted, Q is a maximal parabolic subgroup of G and we are in one of the situations classified by Onishchik (see [O1, O2]). There are two series of flag manifolds Z where this is possible and one additional isolated example:

- Odd dimensional projective spaces where $\tilde{G} = \text{SL}_{2n}(\mathbb{C})$ and $G = \text{Sp}_n(\mathbb{C})$.
- The space of isotropic n -planes with respect to the standard complex bilinear form on \mathbb{C}^{2n} where $\tilde{G} = \text{SO}_{2n}(\mathbb{C})$ and $G = \text{SO}_{2n-1}(\mathbb{C})$.
- The 5-dimensional quadric where $\tilde{G} = \text{SO}_7(\mathbb{C})$ and $G = G_2$.

Example. Let V be a complex vector space and equip $W = V \oplus V^*$ with its standard symplectic form which is defined by

$$\omega(v + \varphi, v' + \varphi') = \varphi'(v) - \varphi(v').$$

Let $\{e_1, \dots, e_n\}$ be a basis of V . Translating to the numerical space, for $z, w \in \mathbb{C}^{2n}$ it follows that $\omega(z, w) = z^t J w$ where

$$J := \begin{pmatrix} 0 & -\text{Id} \\ \text{Id} & 0 \end{pmatrix}.$$

Define the Hermitian form h on \mathbb{C}^{2n} by

$$h(z, w) = \frac{i}{2} \bar{z}^t J w.$$

It is of signature (n, n) . For example a maximal negative (resp. positive) space is spanned by vectors of the form (a, ia) (resp. $(a, -ia)$).

Let $\tilde{G} := \mathrm{SL}_{\mathbb{C}}(W)$ and $G := \mathrm{Sp}_{\mathbb{C}}(W, \omega)$. Define \tilde{G}_0 to be the subgroup of h -isometries in \tilde{G} and $G_0 = \tilde{G}_0 \cap G$. Observe that $\tilde{G}_0 \cong \mathrm{SU}(n, n)$ and $G_0 \cong \mathrm{Sp}_{2n}(\mathbb{R})$.

One directly checks that \tilde{G}_0 has exactly three orbits in $Z = \mathbb{P}(W)$, namely the spaces D^+ and D^- of positive (resp. negative) (complex) lines and the real hypersurface Σ of isotropic lines.

To determine the orbit structure of the G_0 -action is convenient to use Matsuki duality. For this we choose the maximal compact subgroup K_0 of G_0 to be the copy of the unitary group U_n which acts diagonally on W , i.e., $k(v + \varphi) = k(v) + k(\varphi)$. The complexification K of K_0 in G is just $\mathrm{GL}_n(\mathbb{C})$. The duality theorem states that there is a natural bijective correspondence μ between the K -orbits and the G_0 -orbits in Z . This sends a G_0 -orbit \mathcal{O} to the K -orbit $\mu(\mathcal{O}) = K.p$ of any point $p \in \mathcal{O}$ with $K_0.p$ the minimal K_0 -orbit in \mathcal{O} . Furthermore, $\mu(\mathcal{O}) \cap \mathcal{O}$ is just the minimal K_0 -orbit $K_0.p$.

We mention this “duality”, because it is often easier to understand the K -orbit structure than it is to understand the G_0 -orbit structure. In this case one checks directly that K has four orbits in Z , namely the closed orbits $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, a 1-codimensional orbit with two ends whose closure Y consists of the orbit together with the projective subspaces $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, and the complement $Z \setminus Y$ which is the unique open K -orbit. Duality implies that the closed K -orbits are the base cycles in the open G_0 -orbits. Thus G_0 has two open orbits which are of course contained in the two open \tilde{G}_0 -orbits D^+ and D^- . The smaller group G_0 stabilizes the closed \tilde{G}_0 -orbit Σ . There the real points Σ_0 are G_0 -invariant. Thus G_0 has at least two orbits in Σ . But altogether it has only four orbits and therefore we have accounted for all of them: D^+ , D^- , Σ_0 and the complement $\Sigma \setminus \Sigma_0$. In particular, D^+ and D^- are exceptional in the sense that the smaller real form G_0 acts transitively. \square

It would be of interest to have a complete list of all such exceptional cases, in particular to determine if and when a given G_0 -flag domain is *properly* contained in the corresponding \tilde{G}_0 -flag domain. Arguing as above one could possibly compile such a list. This would, however, seem to require a somewhat involved case-by-case discussion which would not be appropriate for the present paper.

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